MATH 135 Sample Midterm F19 (based on F18 and W19 Midterms)

Question 1 (5 points)
(a) Complete the truth table below.

P	Q	$P \Leftrightarrow Q$	$(P \Leftrightarrow Q) \vee Q$	$P \Rightarrow Q$	$(P \Rightarrow Q) \wedge Q$
T	T				
T	F				
F	T				
F	F				

(b) Is $(P \Leftrightarrow Q) \vee Q$ logically equivalent to $(P \Rightarrow Q) \wedge Q$? Circle the correct answer. No further justification is needed.
Equivalent   Not Equivalent

Question 2 (7 points)
Let $x, y \in \mathbb{R}$. Consider the implication S:

$$\text{If } x^2 - y^2 < 0, \text{ then } x < y \text{ or } x + y < 0.$$

(a) State the hypothesis of S.
(b) State the conclusion of S.
(c) State the converse of S.
(d) State the contrapositive of S.
(e) State the negation of S in a form that does not contain an implication.
(f) Prove that for all $x, y \in \mathbb{R}$, if $x^2 - y^2 < 0$, then $x < y$ or $x + y < 0$.

Question 3 (4 points)
(a) Find the coefficient of xy^2 in the expansion of $(3x + y^2)^{10}$.
(b) Let $n \in \mathbb{N}$. Evaluate the sum $\displaystyle\sum_{i=0}^{n} \binom{n}{i} (-1)^{i+1}$.

Question 4 (5 points)
Prove that for all integers $n \geq 3$,

$$\sum_{i=2}^{n-1} \binom{i}{2} = \binom{n}{3}.$$

Question 5 (5 points)
The *Fibonacci Sequence* is defined as follows: $a_1 = 1$, $a_2 = 1$ and $a_m = a_{m-1} + a_{m-2}$ for all integers $m \geq 3$. Prove that for every $n \in \mathbb{N}$, $3 \mid a_{4n}$.

Question 6 (4 points)
Given a variable x, let $P(x)$ denote the open sentence that x is of the form $(4k+1)$ for some $k \in \mathbb{Z}$, and let $Q(x)$ denote the open sentence that x is of the form $(4k+3)$ for some $k \in \mathbb{Z}$. Determine if the following statements are true or false. Circle the correct answers. No justification is needed.

(a) $\forall x \in \mathbb{Z}, (P(x) \vee Q(x))$

True False

(b) $(\forall x \in \mathbb{Z}, P(x)) \vee (\forall x \in \mathbb{Z}, Q(x))$

True False

Let S denote the set of odd integers.

(c) $\forall x \in S, (P(x) \vee Q(x))$

True False

(d) $(\forall x \in S, P(x)) \vee (\forall x \in S, Q(x))$

True False

Question 7 (5 points)
For each of the following statements indicate clearly whether the statement is true or false and then prove or disprove the statement.

(a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, (2x - y)(x + y) = 0.$

Circle the correct answer: True False

(b) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, (2x - y)(x + y) = 0.$

Circle the correct answer: True False

Question 8 (5 points)
Prove that for all $x \in \mathbb{R}$, if x is irrational, then $\sqrt{x}$ is irrational.

Question 9 (5 points)
Let $a \in \mathbb{Z}$. Prove that $a \neq 0$ if and only if there exists $b \in \mathbb{Z}$ with $b \geq 1$ such that $a \mid b$.

Question 10 (5 points)
Prove that there is no solution to $4x^3 - y^2 = 1$ where x and y are integers.